Concepts of Programming Language Design

# Inference Rule Exercises

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1. **Strange Loops:** The following system, based on a system called MIU, is perhaps famously mentioned in Douglas Hofstadter's book, Gödel, Escher, Bach (see, for example, "MU puzzle" on Wikipedia).

$$\frac{}{\texttt{MI}\ \mathbf{MIU}} \qquad (\textit{MIU-1})$$

$$\frac{x\texttt{I}\ \mathbf{MIU}}{x\texttt{IU}\ \mathbf{MIU}} \qquad (\textit{MIU-2})$$

$$\frac{\texttt{M}x\ \mathbf{MIU}}{\texttt{M}xx\ \mathbf{MIU}} \qquad (\textit{MIU-3})$$

$$\frac{x\texttt{III}y\ \mathbf{MIU}}{x\texttt{U}y\ \mathbf{MIU}} \qquad (\textit{MIU-4})$$

$$\frac{x\texttt{UU}y\ \mathbf{MIU}}{xy\ \mathbf{MIU}} \qquad (\textit{MIU-5})$$

   (a) Is $\texttt{MUII}\ \mathbf{MIU}$ derivable? If so, show the derivation tree. If not, explain why not.

**Solution:**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{MI}\ \mathbf{MIU}}\,1}{\texttt{MII}\ \mathbf{MIU}}\,3}{\texttt{MIIII}\ \mathbf{MIU}}\,3}{\texttt{MIIIIIIII}\ \mathbf{MIU}}\,3}{\texttt{MIIIIIIIIU}\ \mathbf{MIU}}\,2}{\texttt{MIIIIIUU}\ \mathbf{MIU}}\,4}{\texttt{MIIIII}\ \mathbf{MIU}}\,5}{\texttt{MUII}\ \mathbf{MIU}}\,4$$

   (b) Is $\dfrac{x\texttt{IU}\ \mathbf{MIU}}{x\texttt{I}\ \mathbf{MIU}}$ admissible? Is it derivable? Justify your answer.

**Solution:** It is not derivable, but it is admissible. It is not derivable because there is no way to construct a tree that looks like this:

$$\dfrac{\dfrac{x\texttt{IU}\ \mathbf{MIU}}{\vdots}}{x\texttt{I}\ \mathbf{MIU}}$$

It is, however, admissible because it does not change the language MIU. There is no string $x$ that could be judged $x$ $\mathbf{MIU}$ with this rule that could not be so judged without it. We could show this by proving the rule using rule induction.

   (c) **Tricky:** Perhaps famously, $\texttt{MU}\ \mathbf{MIU}$ is not admissible. Prove this using rule induction. Hint: Try proving something related to the number of Is in the string.

**Solution:** We will prove that the number of $\texttt{I}$s in any string in MIU is not divisible by three. Seeing as $\texttt{MU}$ has zero $\texttt{I}$s (a multiple of three), if we prove the above, we prove that $\texttt{MU}$ is not admissible.

*Base Case (From rule 1).* We see that the string $\texttt{MI}$ has only one $\texttt{I}$, which is not a multiple of three, hence we have shown our goal.

Inductive case (From rule 2). Given that the number of $\texttt{I}$s in $x\texttt{I}$ is not divisible by three (our inductive hypothesis), we can easily see that the number of $\texttt{I}$s in $x\texttt{IU}$ is identical and therefore is similarly not divisible by three.

Inductive case (From rule 3). Let $n$ be the number of $\texttt{I}$s in $\texttt{M}x$. Our inductive hypothesis is that $3 \nmid n$. The number of $\texttt{I}$s in $\texttt{M}xx$, clearly $2n$, is similarly indivisible, i.e $3 \nmid n \implies 3 \nmid 2n$.

Inductive case (From rule 4). Let $n$ be the number of $\texttt{I}$s in $x\texttt{III}y$. Our inductive hypothesis is that $3 \nmid n$. The number of $\texttt{I}$s in $x\texttt{U}y$, clearly $n-3$, is similarly indivisible, i.e $3 \nmid n \implies 3 \nmid (n-3)$

Inductive case (From rule 5). Given that the number of $\texttt{I}$s in $x\texttt{UU}y$ is the same as the number of $\texttt{I}$s in $xy$, our inductive hypothesis trivially proves our goal.

Thus, by induction, no string in MIU has a number of $\texttt{I}$s divisible by three. Therefore, $\texttt{MU}$ **MIU** is not admissible. $\square$

(d) Here is another language, which we'll call MI:

$$\frac{}{\texttt{MI}\ \textbf{MI}} \quad (\textit{MI-1})$$

$$\frac{\texttt{M}x\ \textbf{MI}}{\texttt{M}xx\ \textbf{MI}} \quad (\textit{MI-2})$$

$$\frac{x\texttt{IIIIII}y\ \textbf{MI}}{xy\ \textbf{MI}} \quad (\textit{MI-3})$$

i. Prove using rule induction that all strings in MI could be expressed as follows, for some $k$ and some $i$, where $2^k - 6i > 0$ (where $\texttt{C}^n$ is the character $\texttt{C}$ repeated $n$ times):

$$\texttt{M}\,\texttt{I}^{2^k-6i}$$

**Solution:**

*Base case (From rule A).* $\texttt{MI} = \texttt{M}\,\texttt{I}^{2^k-6i}$ when $2^k - 6i = 1$, i.e when $k = 0$ and $i = 0$.

Inductive case (From rule B) Given that $\texttt{M}x = \texttt{MI}^{2^a-6b}$ (our inductive hypothesis), we must show that $\texttt{M}xx = \texttt{M}\,\texttt{I}^{2^k-6i}$ for some $k$ and some $i$. As $x = \texttt{I}^{2^a-6b}$ (from I.H), it is easy to see that $xx = \texttt{I}^{2(2^a-6b)} = \texttt{I}^{2^{a+1}-6(2b)} = \texttt{I}^{2^k-6i}$ for $k = a+1$ and $i = 2b$.

Inductive case (from rule C) Given that $x\texttt{IIIIII}y = \texttt{M}\,\texttt{I}^{2^a-6b}$ (our inductive hypothesis). We must show that $xy = \texttt{M}\,\texttt{I}^{2^k-6i}$ for some $k$ and $i$. It should be clear to see that this rule simply subtracts six $\texttt{I}$ characters, and therefore $xy = \texttt{MI}^{2^a-6(b+1)}$, hence $k = a$ and $i = b+1$.

Thus, all strings in MI can be expressed as $\texttt{M}\,\texttt{I}^{2^k-6i}$ where $2^k - 6i > 0$ $\square$

ii. We will now prove the opposite claim that, for all $k$ and $i$, assuming $2^k - 6i > 0$:

$$\texttt{M}\,\texttt{I}^{2^k-6i}\ \textbf{MI}$$

To prove this we will need a few lemmas which we will prove separately.

$\alpha$) Prove, using induction on the natural number $k$ (i.e when $k = 0$ and when $k = k' + 1$), that $\texttt{M}\,\texttt{I}^{2^k}$ **MI**

**Solution:**

*Base case (when $k = 0$).* We have to show $\texttt{MI}$ **MI**, which is true by rule A.
Inductive case (when $k = k' + 1$) We have to show $\texttt{MI}^{2^{k'+1}}$ **MI**, with the inductive hypothesis that $\texttt{MI}^{2^{k'}}$ **MI**. Equivalently, we have to show $\texttt{MI}^{2^{k'}}\texttt{I}^{2^{k'}}$ **MI**, as follows:

$$\dfrac{\dfrac{}{\texttt{MI}^{2^{k'}}\ \textbf{MI}}\,I.H}{\texttt{MI}^{2^{k'}}\texttt{I}^{2^{k'}}\ \textbf{MI}}\,B$$

Therefore, by induction on the natural number $k$, we have shown $\forall k. \texttt{M}\,\texttt{I}^{2^k}$ **MI**. □

$\beta$) Prove, using induction on the natural number $i$, that $\texttt{M}\,\texttt{I}^k$ **MI** implies $\texttt{M}\,\texttt{I}^{k-6i}$ **MI**, assuming $k - 6i > 0$.

**Solution:**

*Base case (when $i = 0$).* We must show that $\texttt{M}\,\texttt{I}^k$ **MI** implies $\texttt{M}\,\texttt{I}^{k-0}$ **MI**, which is obviously a tautology.
Inductive case (when $i = i' + 1$) We must show that $\texttt{M}\,\texttt{I}^k$ **MI** implies $\texttt{M}\,\texttt{I}^{k-6(i'+1)}$ **MI**, given the inductive hypothesis that $\texttt{M}\,\texttt{I}^{k-6i'}$ **MI**. Note that our I.H can be restated as $\texttt{MIIIIII}\,\texttt{I}^{k-6(i'+1)}$ **MI** due to our assumption that $k - 6(i' + 1) > 0$. With this, we can prove our goal as shown:

$$\dfrac{\dfrac{}{\texttt{MIIIIII}\,\texttt{I}^{k-6(i'+1)}\ \textbf{MI}}\,I.H}{\texttt{M}\,\texttt{I}^{k-6(i'+1)}\ \textbf{MI}}\,C$$

□

Therefore, our goal is shown by induction.

Hence, as we know $\texttt{M}\,\texttt{I}^{2^k}$ **MI** for all $k$ from lemma $\alpha$, we can conclude from lemma $\beta$ that $\texttt{M}\,\texttt{I}^{2^k-6i}$ **MI** for all $k$ and all $i$ where $2^k - 6i > 0$ by modus ponens.

These two parts prove that the language MI is exactly characterised by the formulation $\texttt{M}\,\texttt{I}^{2^k-6i}$ where $2^k - 6i > 0$. A very useful result!

iii. Hence prove or disprove that the following rule is admissible in MI:

$$\frac{\texttt{M}x x\ \textbf{MI}}{\texttt{M}x\ \textbf{MI}} \qquad (\text{LEMMA}_1)$$

**Solution:** We know from part i that $\texttt{M}xx$ **MI** $\implies x^2 = \texttt{I}^{2^k-6i}$ for some $k$ and some $i$ where $2^k - 6i > 0$.
This rule is not admissible as it adds strings to the language. As $2^4 - 6 = 10$, we know $\texttt{MI}^{10}$ is in the language. This rule would make $\texttt{MI}^5$ a string in the language which it is not as there is no $k$ and $i$ such that $2^k - 6i = 5$.

iv. Why is the following rule **not** admissible in MI?

$$\frac{xy\ \textbf{MI}}{x\texttt{IIIIII}y\ \textbf{MI}} \qquad (\text{LEMMA}_2)$$

**Solution:** The rule is not admissible as it adds strings to the language. This allows us to add six $\texttt{I}$ characters to any string in MI and judge it in MI, which results in additional strings. For example, applying the rule to $\texttt{MI}$ (which is in MI), gives us $\texttt{M}\,\texttt{I}^7$, when our existing formulation of MI($\texttt{M}\,\texttt{I}^{2^k-6i}$) clearly only allows for even amounts of $\texttt{I}$s.

v. Prove that, for all $s$, $s$ **MI** $\implies$ $s$ **MIU**. You can prove it using the characterisation we have already developed, or directly by induction.

**Solution:**

**Using the characterisation** We shall show that all strings in MI, characterised by $\mathtt{M}\,\mathtt{I}^{2^k-6i}$ where $2^k - 6i > 0$, are also in MIU. That is, we shall show that $\mathtt{M}\,\mathtt{I}^{2^k-6i}$ **MIU**.
To start, we shall prove inductively on $k$ that $\mathtt{M}\,\mathtt{I}^{2^k}$ **MIU** for all $k$.

*Base case (Where $k = 0$).* We must show $\mathtt{MI}$ **MIU**, which we know trivially from rule 1.
Inductive case (where $k = k'+1$). We must show $\mathtt{M}\,\mathtt{I}^{2^{k'+1}}$ **MIU**, given the inductive hypothesis that $\mathtt{M}\,\mathtt{I}^{2^{k'}}$ **MIU**. Note we can restate our proof goal as $\mathtt{M}\,\mathtt{I}^{2^{k'}}\,\mathtt{I}^{2^{k'}}$ **MIU**

$$\dfrac{\dfrac{}{\mathtt{M}\,\mathtt{I}^{2^{k'}}\ \mathbf{MIU}}\,I.H}{\mathtt{M}\,\mathtt{I}^{2^{k'}}\,\mathtt{I}^{2^{k'}}\ \mathbf{MIU}}\,B$$

Thus we have shown by induction that $\mathtt{M}\,\mathtt{I}^{2^k}$ **MIU** for all $k$.

Next we must prove that $\mathtt{M}\,\mathtt{I}^{k}$ **MIU** implies $\mathtt{M}\,\mathtt{I}^{k-6i}$ **MIU** for all $i$, assuming $k - 6i > 0$.

Base case (where $i = 0$), we must show that $\mathtt{M}\,\mathtt{I}^{k}$ **MIU** implies $\mathtt{M}\,\mathtt{I}^{k-0}$ **MIU**, which is trivially a tautology.
Inductive case (where $i = i' + 1$) we must show that $\mathtt{M}\,\mathtt{I}^{k-6(i'+1)}$ **MIU** given the inductive hypothesis $\mathtt{M}\,\mathtt{I}^{k-6i'}$ **MIU**. As we know $k-6(i'+1) > 0$, we can restate our inductive hypothesis as $\mathtt{MIIIIII}\,\mathtt{I}^{k-6(i'+1)}$ **MIU**, and easily prove our goal:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathtt{MIIIIII}\,\mathtt{I}^{k-6(i'+1)}\ \mathbf{MIU}}\,I.H}{\mathtt{MUIII}\,\mathtt{I}^{k-6(i'+1)}\ \mathbf{MIU}}\,4}{\mathtt{MUU}\,\mathtt{I}^{k-6(i'+1)}\ \mathbf{MIU}}\,4}{\mathtt{M}\,\mathtt{I}^{k-6(i'+1)}\ \mathbf{MIU}}\,5$$

Thus, by modus ponens we can see that $\mathtt{M}\,\mathtt{I}^{2^k-6i}$ **MIU** for all $k$ and $i$ where $2^k - 6i > 0$. As this is the exact characterisation of MI, we have proven that $s$ **MI** $\implies$ $s$ **MIU** for all $s$. $\square$

**Using straight forward induction** (solution provided by Ramón Rico Cuevas)

- Base case: $s$ is in MIvia rule (A), then $s = \mathtt{MI}$ also in MIU via rule 1.
- Inductive case 1: $s$ is in MIvia rule (B), then $\exists x.s = \mathtt{M}xx$, with
  - (A1) $\mathtt{M}x$ **Mi**
  - (IH) $\mathtt{M}x$ **MI** implies $\mathtt{M}x$ **MIU**

$$\dfrac{\dfrac{\dfrac{}{\mathtt{M}x\ \mathbf{MI}}\,A1}{\mathtt{M}x\ \mathbf{MIU}}\,IH}{\mathtt{M}xx\ \mathbf{MIU}}\,3$$

- Inductive case 2: $s$ is in MIvia rule (B), then $\exists x.y.s = xy$ with
  - (A1)$x\mathtt{IIIIII}y$ **MI**
  - (IH) $x\mathtt{IIIIII}y$ **MI** implies $x\mathtt{IIIIII}y$ **MIU**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x\mathtt{IIIIII}y\ \mathbf{MI}}\,A1}{x\mathtt{IIIIII}y\ \mathbf{MIU}}\,IH}{x\mathtt{UIII}y\ \mathbf{MIU}}\,4}{x\mathtt{UU}y\ \mathbf{MIU}}\,4}{xy\ \mathbf{MIU}}\,5$$

2. **Counting Sticks:** The following language (also presented in a similar form by Douglas Hofstadter, but the original invention is not his) is called the $\Phi\Psi$ system. Unlike the MIU language discussed above, this

language is not comprised of a single judgement, but of a ternary relation, written $x \; \Phi \; y \; \Psi \; z$, where $x$, $y$ and $z$ are strings of hyphens (i.e '-'), which may be empty ($\epsilon$). The system is defined as follows:

$$\frac{}{\epsilon \; \Phi \; x \Psi \; x} \qquad (\Phi\Psi - 1)$$

$$\frac{x \; \Phi \; y \; \Psi \; z}{\texttt{-}x \; \Phi \; y \; \Psi \; \texttt{-}z} \qquad (\Phi\Psi - 2)$$

(a) Prove that $\texttt{--} \; \Phi \; \texttt{---} \; \Psi \; \texttt{-----}$.

> **Solution:**
>
> $$\dfrac{\dfrac{\dfrac{}{\epsilon \; \Phi \; \texttt{---} \; \Psi \; \texttt{---}}\,B}{\texttt{-} \; \Phi \; \texttt{---} \; \Psi \; \texttt{----}}\,I}{\texttt{--} \; \Phi \; \texttt{---} \; \Psi \; \texttt{-----}}\,I$$

(b) Is the following rule admissible? Is it derivable? Explain your answer

$$\frac{\texttt{-}x \; \Phi \; y \; \Psi \; \texttt{-}z}{x \; \Phi \; y \; \Psi \; z} \qquad (\Phi\Psi - 2')$$

> **Solution:** It is not derivable (as it cannot be shown with a proof tree), but it is admissible. We know this because the language definition for $\Phi\Psi$ is unambiguous, so the only way for $\texttt{-}x \; \Phi \; y \; \Psi \; \texttt{-}z$ to hold is if this was established by rule $I$. Therefore, we can deduce that $x \; \Phi \; y \; \Psi \; z$, as this is the premise of rule $I$. We can often "flip" or invert rules in this way, but only if the language definition is unambiguous.

(c) Show that $x \; \Phi \; \epsilon \; \Psi \; x$, for all hyphen strings $x$, by doing induction on the length of the hyphen string (where $x = \epsilon$ and $x = \texttt{-}x'$).

> **Solution:**
>
> *Base case (where $x = \epsilon$).* We must show that $\epsilon \; \Phi \; \epsilon \; \Psi \; \epsilon$, which is trivially true by rule $B$.
>
> Inductive case (where $x = \texttt{-}x'$) We have the inductive hypothesis $x' \; \Phi \; \epsilon \; \Psi \; x'$, and must show that $\texttt{-}x' \; \Phi \; \epsilon \; \Psi \; \texttt{-}x'$. Our goal trivially reduces to our induction hypothesis by rule $I$.
>
> Therefore we have shown $x \; \Phi \; \epsilon \; \Psi \; x$ for all $x$ by induction on $x$. □

(d) Show that if $\texttt{-}x \; \Phi \; y \; \Psi \; z$ then $x \; \Phi \; \texttt{-}y \; \Psi \; z$, for all hyphen strings $x$, $y$ and $z$, by doing a rule induction on the premise.

> **Solution:**
>
> *Base case.* (From rules B and I, where $\texttt{-} \; \Phi \; y \; \Psi \; \texttt{-}y$). We must show that $\epsilon \; \Phi \; \texttt{-}y \; \Psi \; \texttt{-}y$, which is trivially true by rule $B$.
>
> Inductive case. (From rule I, where $\texttt{--}x' \; \Phi \; y \; \Psi \; \texttt{-}z'$ $(*)$). We have the inductive hypothesis:
>
> $$\frac{\texttt{-}x' \; \Phi \; y \; \Psi \; z'}{x' \; \Phi \; \texttt{-}y \; \Psi \; z'}\,I.H$$
>
> We must show that $\texttt{-}x' \; \Phi \; \texttt{-}y \; \Psi \; \texttt{-}z'$.
>
> $$\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{--}x' \; \Phi \; y \; \Psi \; \texttt{-}z'}\,(*)}{\texttt{-}x' \; \Phi \; y \; \Psi \; z'}\,I'}{x' \; \Phi \; \texttt{-}y \; \Psi \; z'}\,I.H}{\texttt{-}x' \; \Phi \; \texttt{-}y \; \Psi \; \texttt{-}z'}\,I$$
>
> Thus we have shown by induction that if $\texttt{-}x \; \Phi \; y \; \Psi \; z$ then $x \; \Phi \; \texttt{-}y \; \Psi \; z$, for all hyphen strings $x$, $y$ and $z$.
>
> □

(e) Show that $x \ \Phi \ y \ \Psi \ z$ implies $y \ \Phi \ x \ \Psi \ z$.

**Solution:** We show this by rule induction on the premise with the rules of $\Phi\Psi$.

*Base case.* (From rule B, where $\epsilon \ \Phi \ y \ \Psi \ y$). We must show that $y \ \Phi \ \epsilon \ \Psi \ y$. We proved this, most fortunately, above in part (c).

Inductive case. (From rule I, where $\texttt{-}x' \ \Phi \ y \ \Psi \ -z'$ $(*)$ ). We have the inductive hypothesis:

$$\frac{x' \ \Phi \ y \ \Psi \ z'}{y \ \Phi \ x' \ \Psi \ z'} I.H$$

We must show that $y \ \Phi \ \texttt{-}x' \ \Psi \ \texttt{-}z'$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\overline{\texttt{-}x' \ \Phi \ y \ \Psi \ -z'}\,(*)}{x' \ \Phi \ y \ \Psi \ z'} I'}{y \ \Phi \ x' \ \Psi \ z'} I.H}{\texttt{-}y \ \Phi \ x' \ \Psi \ \texttt{-}z'} I}{y \ \Phi \ \texttt{-}x' \ \Psi \ \texttt{-}z'} (d)$$

Thus we have shown by induction that $x \ \Phi \ y \ \Psi \ z$ implies $y \ \Phi \ x \ \Psi \ z$. $\square$

(f) Have you figured out what the $\Phi\Psi$ system actually is? Prove that if $\texttt{-}^x \ \Phi \ \texttt{-}^y \ \Psi \ \texttt{-}^z$, then $z = \texttt{-}^{x+y}$ (where $\texttt{-}^x$ is a hyphen string of length $x$).

**Solution:** We proceed by rule induction on the premise.

*Base case.* (From rule $B$, where $\texttt{-}^0 \ \Phi \ \texttt{-}^y \ \Psi \ \texttt{-}^y$), we must show that $\texttt{-}^0 \ \Phi \ \texttt{-}^y \ \Psi \ \texttt{-}^{0+y}$. As $0 + y = y$, this trivially reduces to the premise.

Inductive case (From rule $I$, where $\texttt{-}^{x'+1} \ \Phi \ \texttt{-}^y \ \Psi \ \texttt{-}^{z'+1}$ $(*)$), we have the inductive hypothesis that $\texttt{-}^{x'} \ \Phi \ \texttt{-}^y \ \Psi \ \texttt{-}^{z'} \implies z' = x' + y$. We must show that $z' + 1 = (x' + 1) + y$, or, equivalently, that $z' = x' + y$:

$$\cfrac{\cfrac{\overline{\texttt{-}^{x'+1} \ \Phi \ \texttt{-}^y \ \Psi \ \texttt{-}^{z'+1}}\,(*)}{\texttt{-}^{x'} \ \Phi \ \texttt{-}^y \ \Psi \ \texttt{-}^{z'}} I}{z' = x' + y} I.H$$

Thus we have shown by rule induction that the $\Phi\Psi$ system is in fact unary addition. $\square$

3. **Ambiguity and Simultaneity**: Here is a simple grammar for a functional programming language [1]:

$$\frac{x \in \mathbb{N}}{x \ \textbf{\textit{Expr}}} \tag{E-1}$$

$$\frac{e_1 \ \textbf{\textit{Expr}} \quad e_2 \ \textbf{\textit{Expr}}}{e_1 e_2 \ \textbf{\textit{Expr}}} \tag{E-2}$$

$$\frac{e \ \textbf{\textit{Expr}}}{\lambda e \ \textbf{\textit{Expr}}} \tag{E-3}$$

$$\frac{e \ \textbf{\textit{Expr}}}{(e) \ \textbf{\textit{Expr}}} \tag{E-4}$$

(a) Is this grammar ambiguous? If not, explain why not. If so, give an example of an expression that has multiple parse trees.

[1] if you're interested, it's called lambda calculus, with de Bruijn indices syntax, not that it's relevant to the question!

**Solution:** Yes, the expression 1 2 3 could be parsed two different ways, i.e:

$$\dfrac{\dfrac{\dfrac{}{1\ \textbf{\textit{Expr}}}\textsc{Var.}\quad \dfrac{}{2\ \textbf{\textit{Expr}}}\textsc{Var.}}{1\ 2\ \textbf{\textit{Expr}}}\textsc{Appl.}\quad \dfrac{}{3\ \textbf{\textit{Expr}}}\textsc{Var.}}{1\ 2\ 3\ \textbf{\textit{Expr}}}\textsc{Appl.}$$

Or:

$$\dfrac{\dfrac{}{1\ \textbf{\textit{Expr}}}\textsc{Var.}\quad \dfrac{\dfrac{}{2\ \textbf{\textit{Expr}}}\textsc{Var.}\quad \dfrac{}{3\ \textbf{\textit{Expr}}}\textsc{Var.}}{2\ 3\ \textbf{\textit{Expr}}}\textsc{Appl.}}{1\ 2\ 3\ \textbf{\textit{Expr}}}\textsc{Appl.}$$

(b) Develop a new (unambiguous) grammar that encodes the left associativity of application, that is `1 2 3 4` should be parsed as `((1 2) 3) 4` (modulo parentheses). Furthermore, lambda expressions should extend as far as possible, i.e $\lambda$ `1 2` is equivalent to $\lambda$ `(1 2)` not `(`$\lambda$ `1) 2`.

**Solution:**

$$\dfrac{x \in \mathbb{N}}{x\ \textbf{\textit{AExpr}}}\text{AVar.}\qquad \dfrac{e_1\ \textbf{\textit{PExpr}}\quad e_2\ \textbf{\textit{AExpr}}}{e_1 e_2\ \textbf{\textit{PExpr}}}\text{AAppl.}\qquad \dfrac{e\ \textbf{\textit{LExpr}}}{\lambda e\ \textbf{\textit{LExpr}}}\text{AAbs.}$$

$$\dfrac{e\ \textbf{\textit{LExpr}}}{(e)\ \textbf{\textit{AExpr}}}\text{AParen.}y\qquad \dfrac{e\ \textbf{\textit{PExpr}}}{e\ \textbf{\textit{LExpr}}}\textsc{Shunt}_1\qquad \dfrac{e\ \textbf{\textit{AExpr}}}{e\ \textbf{\textit{PExpr}}}\textsc{Shunt}_2$$

(c) **Tricky** Prove that all expressions in your grammar are representable in ***Expr***, that is, that your grammar describes only strings that are in ***Expr***.

**Solution:** We shall prove the following simultaneously:

- $x\ \textbf{\textit{LExpr}} \Rightarrow x\ \textbf{\textit{Expr}}$
- $x\ \textbf{\textit{PExpr}} \Rightarrow x\ \textbf{\textit{Expr}}$
- $x\ \textbf{\textit{AExpr}} \Rightarrow x\ \textbf{\textit{Expr}}$

*Proof.* Base case (From rule AVar., where $x$ ***AExpr*** for some $x \in \mathbb{N}$). We must show $x$ ***Expr***, trivial by rule Var.

Inductive case. (From rule AAppl., where $e_1 e_2$ ***PExpr***. By inversion on rule AAppl., we deduce $e_1$ ***PExpr*** (∗), and $e_2$ ***AExpr*** (∗∗). We have the inductive hypotheses $e_1\ \textbf{\textit{AExpr}} \vee e_1\ \textbf{\textit{PExpr}} \vee e_1\ \textbf{\textit{LExpr}} \implies e_1\ \textbf{\textit{Expr}}$ ($\text{I.H}_1$) and $e_2\ \textbf{\textit{AExpr}} \vee e_2\ \textbf{\textit{PExpr}} \vee e_2\ \textbf{\textit{LExpr}} \implies e_2\ \textbf{\textit{Expr}}$ ($\text{I.H}_2$). By lifting the three disjunctions out of the implications, we get six implications, essentially saying that if $e_1$ or $e_2$ is either $PExpr$, $LExpr$, or $AExpr$, then it is $Expr$.

We must show that $e_1 e_2$ ***Expr***.

$$\dfrac{\dfrac{\dfrac{}{e_1\ \textbf{\textit{LExpr}}}(*)}{e_1\ \textbf{\textit{Expr}}}IH_1\quad \dfrac{\dfrac{}{e_2\ \textbf{\textit{AExpr}}}(**)}{e_2\ \textbf{\textit{Expr}}}IH_2}{e_1 e_2\ \textbf{\textit{Expr}}}$$

Inductive case. (From rule AAbs., where $\lambda x$ ***LExpr***). We can deduce by inversion of rule AAbs. that $x$ ***LExpr***. Applying one of our inductive hypotheses $x\ \textbf{\textit{LExpr}} \implies x\ \textbf{\textit{Expr}}$ to this, we can deduce that $x$ ***Expr***, and then we can apply forwards the rule Abs. to prove our goal: $\lambda x$ ***Expr***.

Inductive case. (From rule AParen., where $(x)$ ***AExpr***). We can deduce by inversion that $x$ ***LExpr***. Using one of the I.H, we get $x$ ***Expr***, then by rule Paren. we show our goal $(x)$ ***Expr***.

The inductive case for the rules $\textsc{Shunt}_1$ and $\textsc{Shunt}_2$ are trivial as they do not alter the expression.

Thus, by induction, $s\ \boldsymbol{LExpr} \vee s\ \boldsymbol{PExpr} \vee s\ \boldsymbol{AExpr} \implies s\ \boldsymbol{Expr}$. We can state this more succinctly thanks to the SHUNT rules as $s\ \boldsymbol{LExpr} \implies s\ \boldsymbol{Expr}$.

$\square$